A component-based data assimilation strategy with application to vascular flow

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Introduction of the problem

The goal of the project is to make a **local state estimation** of a fluid flow in a network.

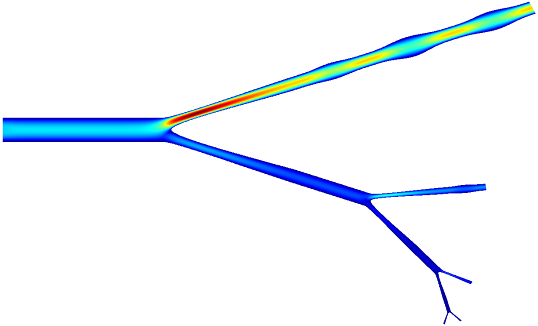


Figure: Example of network

What we know

- the geometry of the domain of interest
- measures of the field on this domain

Settings of the problem

We want to deploy an **offline-online** strategy to make the estimation **as fast as possible**. In this context we want to use the **Parametrized-Background Data Weak** method to combine knowledge given by:

- a parametrized mathematical **model**,
- **measures** of the field.

Limitations

- the model can not catch the exact solution $\Rightarrow$ **non-parametric uncertainty**,
- measures are **indirect** and **noisy**.

Component-based approach

- a library of **components**
- the ability to make **networks**
- an **expensive high-fidelity solver**

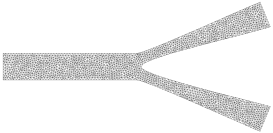


Figure: Bifurcation component

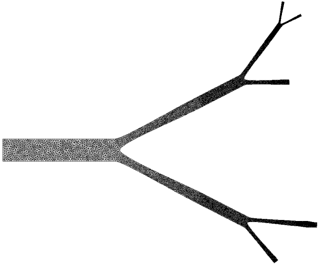


Figure: Example of network

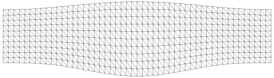


Figure: Channel component

Component-based approach: geometrical deformation

Geometrical deformation

We define a geometrical parameter μ, used to *map* **reference** domain $\hat{\Omega}$ to some **physical** domain Ω_μ:

$$\phi_\mu : \hat{\Omega} \mapsto \phi_\mu(\hat{\Omega}) = \Omega_\mu. \quad (1)$$

This parameter μ belongs to $\mathcal{P} \subset \mathbb{R}^{10}$ and encodes deformation, shift, scaling and rotation.

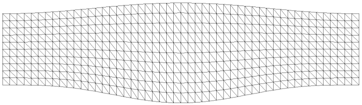


Figure: Channel component

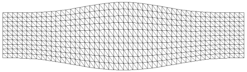


Figure: Deformed component

Component-based approach: random networks

With the parametrized-mapping, we can make random network with N^{co} components.

Using the **high-fidelity solver** we can solve problems associated to the model on these random networks to build **global snapshots** collection.
We then extract solutions on components and map-back to build libraries of **local snapshots**.

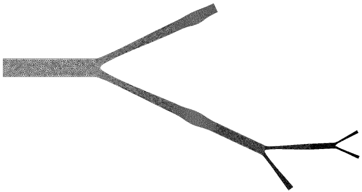


Figure: Example of network

Mathematical model: Navier-Stokes

Steady Navier-Stokes equation

Set $\mu^{\mathrm{net}} \in \mathcal{P}^{\mathrm{net}}$, find $(u(\mu), p(\mu)) \in [H^1(\Omega_\mu)]^2 \times L^2(\Omega_\mu)$ such that :

$$
\begin{cases}
-\nu_\mu \Delta u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \Omega_\mu \\
\nabla \cdot u = 0 & \text{on } \Omega_\mu \\
u = g_\mu & \text{on } \Gamma_{\mathrm{input},\mu} \\
u = 0 & \text{on } \Gamma_{\mathrm{wall},\mu} \\
\nu_\mu \frac{\partial u}{\partial \vec{n}} + p \vec{n} = 0 & \text{on } \Gamma_{\mathrm{output},\mu}
\end{cases}
\tag{2}
$$

- Equations solved using $(\mathbb{P}^k - \mathbb{P}^k)$ finite element, with SUPG stabilization.
- Can be used only on networks.
- ν_μ and g_μ are random.
- Here μ^{net} is global, it contains ν_μ, input parameters and every component deformation.

About measures

We define M *measure functionals* as:

$$l_m(v) = \frac{1}{2\pi\sigma^2} \int_{\Omega_\mu} e^{\frac{1}{2\sigma^2}(x - x_m^{\mathrm{obs}})^2} [\sin(\theta_m), \cos(\theta_m)] \cdot v \qquad \forall v \in \mathcal{X}. \tag{3}$$

Riesz representation theorem

Let $\left(H, (.,.)_H\right)$ be an Hilbert space, for all $l \in H'$ it exists an unique $\tilde{l} \in H$ such that:

$$l(v) = \left(\tilde{l}, v\right)_H \qquad \forall v \in H.$$

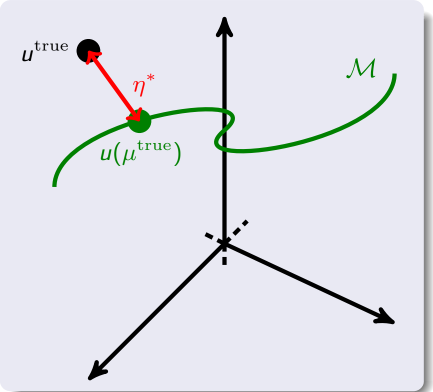
We can than define the **update space** as:

$$\mathcal{U}_M = \mathrm{span}\{\tilde{l}_m\}_{m=1}^M = \mathrm{span}\{q_m\}_{m=1}^M,$$

with q_m an orthonormalized basis.

Intuition about the PBDW method

u^{true} belongs to some Hilbert space $\mathcal{X}$.



The **model** is the manifold span by parametrized PDE:

$$\mathcal{M} = \{u(\mu) \in \mathcal{X},\ \forall \mu \in \mathcal{P}\} \subset \mathcal{X}.$$

$$\eta^* = u^{\text{true}} - u(\mu^{\text{true}})$$

is the **update** and is related to the **non-parametric uncertainty**.

Intuition about the PBDW method

Background space

The **background space** $\mathcal{Z}_N$ is a **N**-sized vectorial space which approximate the manifold:

$$\forall v \in \mathcal{M}, \ \|v - \Pi_{\mathcal{Z}_N} v\|_{\mathcal{X}} \leq \varepsilon_N. \tag{4}$$

Since $u(\mu^{\mathrm{true}}) \in \mathcal{M}$, it exists $\zeta^* \in \mathcal{Z}_{\mathcal{N}}$ such that:

$$\|u(\mu^{\mathrm{true}}) - \zeta^*\|_{\mathcal{X}} \leq \varepsilon_N \tag{5}$$

PBDW method

$$(\zeta^*, \eta^*) = \arg \inf_{\zeta, \eta \in \mathcal{Z}_N \times \mathcal{X}} \xi \|\eta\|_{\mathcal{X}}^2 + \frac{1}{M} \|L_M(\zeta + \eta) - y_M\|_{2,\mathbb{R}^M}^2 \tag{6}$$

Global formulation of the method

Using the *Representer Theorem* it can be proven that $\eta^* \in \mathcal{U}_M$ [1].
Hence we have the following formulation.

PBDW method

$$(\zeta^*, \eta^*) = \arg \inf_{\zeta, \eta \in \mathcal{Z}_N \times \mathcal{U}_M} \xi \|\eta\|_{\mathcal{X}}^2 + \frac{1}{M} \|L_M(\zeta + \eta) - y_M\|_{2,\mathbb{R}^M}^2 \tag{7}$$

Since $\mathcal{Z}_N = \mathrm{span}\{z_n\}_{n=1}^N$ and $\mathcal{U}_M = \mathrm{span}\{q_m\}_{m=1}^M$, it exists $\alpha^* \in \mathbb{R}^N$ and $\beta^* \in \mathbb{R}^M$ s.t.:

$$\zeta^* = \sum_{n=1}^N \alpha_n^* \, z_n, \qquad \eta^* = \sum_{m=1}^M \beta_m^* \, q_m.$$

Algebraic formulation of the method

PBDW method

$$(\alpha^*, \beta^*) = \arg \inf_{\alpha,\beta \in \mathbb{R}^N \times \mathbb{R}^M} \xi \|\beta\|^2_{2,\mathbb{R}^M} + \frac{1}{M} \Bigg\| \underbrace{\sum_{n=1}^{N} L_M(z_n)}_{L_\zeta} \alpha_n + \underbrace{\sum_{m=1}^{M} L_M(q_m)}_{L_\eta} \beta_m - y_M \Bigg\|^2_{2,\mathbb{R}^M} \quad (8)$$

- Quadratic problem of size $N + M$.
- extension: constrain form $a \leq \alpha \leq b$ improve the stability in presence of noise [1].
- Solve using 2-step procedure and constrained minimization solver in *matlab*.

Guarantee some properties of the true state

- We have $\mathcal{X} = [H^1(\Omega_\mu)]^2$.
- True state verifies $u^{\text{true}}|_{\Gamma_{\text{wall}}} = 0$ and $\nabla \cdot u^{\text{true}} = 0$.
- We want to encode these properties of u^{true} first on the **update space**.

$$\mathcal{X}_0 = \{u \in \mathcal{X}, u|_{\Gamma_{\text{wall}}} = 0\}, \quad \mathcal{X}_{0,\text{div}} = \{u \in \mathcal{X}_0, \nabla \cdot u = 0\}$$

- $\mathcal{Z}_N \subset \mathcal{X}_0$, geometry deformation keep homogeneous Dirichlet traces on boundary.
- $\mathcal{Z}_N \not\subset \mathcal{X}_{0,\text{div}}$, geometry deformation change the divergence

Construct incompressible update part

To find $\widetilde{l}_m$ (Riesz representer of l_m in $\mathcal{X}_{0,\mathrm{div}}$), one can solve following problems

Stokes-like problem

Find $(\widetilde{l}_m, p_m) \in \mathcal{X}_0 \times L^2(\Omega)$ s.t. for all $(v, q) \in \mathcal{X}_0 \times L^2(\Omega)$

$$\int_\Omega \widetilde{l}_m \cdot v \; + \int_\Omega \nabla \widetilde{l}_m : \nabla v \; + \int_\Omega (\nabla \cdot v) p_m = l(v)$$

$$\int_\Omega (\nabla \cdot \widetilde{l}_m) q = 0$$

Bound constraint of the divergence

Divergence-constraint PBDW (Ivanov formulation)

$$\begin{cases} \inf_{\zeta,\eta \in \mathcal{Z}_N \times \mathcal{U}_M} \xi \|\eta\|_{\mathcal{X}} + \frac{1}{M} \|L_M(\zeta+\eta) - y_M\|^2_{2,\mathbb{R}^M} \\ \|\nabla \cdot \zeta\|_{L^2(\Omega_\mu)} \leq \varepsilon \end{cases}.$$

Divergence-constraint-PBDW with $\mathcal{U}_M \not\subset \mathcal{X}_{0,\mathrm{div}}$

$$\begin{cases} \inf_{\zeta,\eta \in \mathcal{Z}_N \times \mathcal{U}_M} \xi \|\eta\|_{\mathcal{X}} + \frac{1}{M} \|L_M(\zeta+\eta) - y_M\|^2_{2,\mathbb{R}^M} \\ \|\nabla \cdot (\zeta + \eta)\|_{L^2(\Omega_\mu)} \leq \varepsilon \end{cases}.$$

Limitations

- These formulations have an additional parameter ε to tune.
- Expensive.

Penalization constraint on the divergence

We can enforce the divergence constraint using **Tichonov penalization**.

Divergence-penalized PBDW with $\mathcal{U}_M \subset \mathcal{X}_{0,\mathrm{div}}$

$$\inf_{\zeta,\eta \in \mathcal{Z}_N \times \mathcal{U}_M} \xi \|\eta\|_{\mathcal{X}} + \frac{1}{M} \|L_M(\zeta + \eta) - y_M\|^2_{2,\mathbb{R}^M} + \lambda \|\nabla \cdot \zeta\|^2_{L^2(\Omega_\mu)}$$

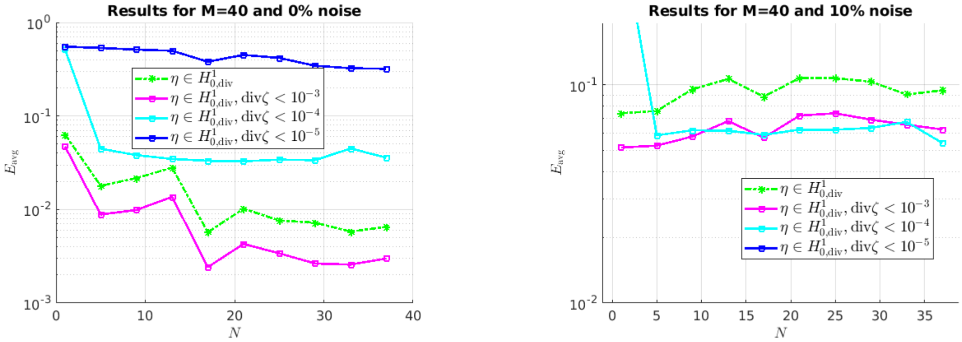
Divergence-penalized PBDW with $\mathcal{U}_M \not\subset \mathcal{X}_{0,\mathrm{div}}$

$$\inf_{\zeta,\eta \in \mathcal{Z}_N \times \mathcal{U}_M} \xi \|\eta\|_{\mathcal{X}} + \frac{1}{M} \|L_M(\zeta + \eta) - y_M\|^2_{2,\mathbb{R}^M} + \lambda \|\nabla \cdot (\zeta + \eta)\|^2_{L^2(\Omega_\mu)}$$

Limitations

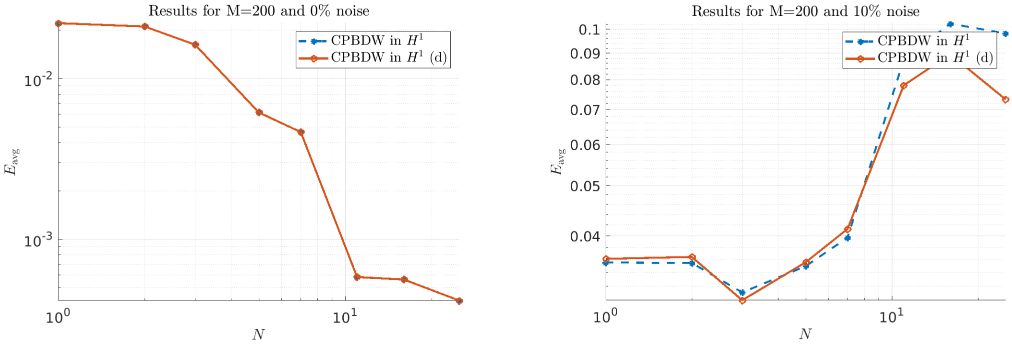
- These formulations have an additional parameter λ to tune.
- Can be expensive.

Divergence-constrained: results PBDW with ($\mathcal{U}_M \subset \mathcal{X}_{0,\text{div}}$)



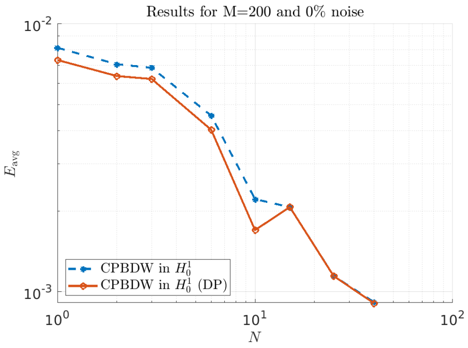
E_{avg}: relative error in L_2 — N_{rep}=30 — $\xi \in [10^{-9}, 10^{9}]$ with cross-validation

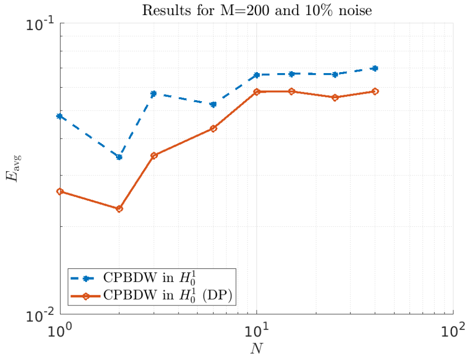
Divergence-penalized: results PBDW with ($\mathcal{U}_M \subset \mathcal{X}_{0,\text{div}}$)



E_{avg}: relative error in L_2 — $N_{\text{rep}}=30$ — $\xi, \lambda \in [10^{-9}, 10^9]$ with cross-validation

Divergence-penalized PBDW: results ($\mathcal{U}_M \subset \mathcal{X}_0$)





$E_{\rm avg}$: relative error in L_2 — $N_{\rm rep}$=30 — $\xi, \lambda \in [10^{-9}, 10^9]$ with cross-validation

Artificial measure functionals

$$\nabla \cdot u^{\mathrm{true}} = 0 \text{ implies: } \forall q \in L^2(\Omega_\mu), \int_{\Omega_\mu} (\nabla \cdot u^{\mathrm{true}}) q = 0$$

Artificial measures functionals

$$l_k(u) = \int_\Omega (\nabla \cdot u) q_k, \quad k = 1, \dots, K.$$

with q_k some function in $L^2(\Omega_\mu)$

- $\forall \eta \in \mathcal{X}_{0,\mathrm{div}}, l_k(\eta) = 0$.
- $l_k(\zeta + \eta) = l_k(\zeta) + l_k(\eta) = l_k(\zeta)$.
- $l_k(u^{\mathrm{true}}) = 0 \Rightarrow$ zero artificial observations.
- $\widetilde{l_k} = 0 \Rightarrow$ unchanged update space.

Artificial measures

Denote

$$M' = M + K, \quad L_{M'}(\zeta + \eta) = \begin{bmatrix} L_M(\zeta + \eta) \\ \\ L_K(\zeta) \end{bmatrix}, \quad y_{M'} = \begin{bmatrix} y_M \\ \\ 0 \end{bmatrix}.$$

Divergence-measured PBDW

$$\inf_{\zeta,\eta \in \mathcal{Z}_N \times \mathcal{U}_M} \xi \|\eta\|_{\mathcal{X}} + \frac{1}{M'} \left\| L_{M'}(\zeta + \eta) - y_{M'} \right\|^2_{2,\mathbb{R}^{M'}}$$

- With $\{q_k\}_{k=1}^K$ a suitable basis of $\mathrm{span}\{\nabla \cdot z_n\}_{n=1}^N$, we get Divergence-penalized PBDW.

Conclusion

Conclusion:

- Improvements are not as impressive as expected ...
- ... maybe some mistakes need to be fixed.
- Several methods have been test separately, they need to be compare.
- Need to find a balanced between cost and improve.

Perspectives:

- Find a way to characterize parameters.
- Improve the training part to reduce the offline cost.

References

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